# Math 733: Vector Fields, Differential Forms, and Cohomology

**Lecture notes**
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Author's note: These are rough lecture notes, very rough, for a course in spring 2010 at Wake Forest University. Comments and corrections are welcome. This is a master's level course that does not assume knowledge of point-set topology; we work primarily with subsets of $\mathbf{R}^3$.

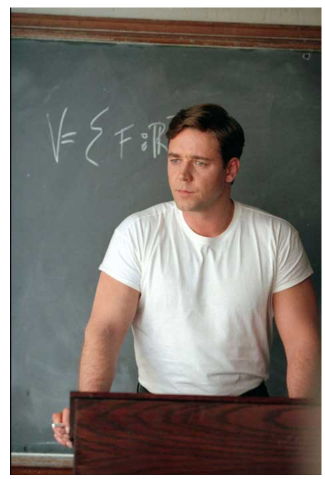
## 1. Introduction – a question from *A Beautiful Mind*

We begin with a question of great interest to many mathematicians, and one that quite a number of vector calculus books state incorrectly. We show a clip from the film *A Beautiful Mind*, which follows the life of John Nash, a brilliant mathematician and the only one (?) to ever win a Nobel Prize. (There's no Nobel Prize in mathematics; the Fields Medal is the equivalent award. Nash won his Nobel in economics for work on game theory.)

*John Nash*

*John Nash*, older

*Crowe*, as Nash



In the film, John Nash (Russell Crowe) is teaching a sophomore-level vector calculus course, um, in his own way. After jettisoning the textbook into the trash, he poses the following problem:

**Problem 1.1** (Nash/Crowe's Problem)**.**

$$\begin{aligned} V =& \{F : (\mathbf{R}^3 \setminus X) \to \mathbf{R}^3 \quad \text{so } \nabla \times F = 0\} \\ W =& \{F = \nabla g\} \qquad\qquad \{\text{'Gradients'}\} \\ & \dim(V \setminus W) = \ ? \end{aligned}$$

Here $X$ is a subset of $\mathbf{R}^3$; $\nabla \times F$ is the *curl* of $F$.

Nash claims this it will take

- "for some of you, many months to solve"
- "for others among you, it will take you *the term of your natural lives*"

Hollywood was being overly dramatic. This is a great problem and relates to my research. But, it's not that hard to solve! We'll have an answer within two months.

The answer actually depends on $X$; it relates to how many different *nontrivial loops* you can draw on the subset $\mathbf{R}^3 \setminus X$.

Let's put this in other terms. Recall that a vector field is really a map $\vec{F} : \Omega \subset \mathbf{R}^3 \to \mathbf{R}^3$ which assigns a vector to every point in the subset $\Omega$; we can write

$$F(x, y, z) = u(x, y, z)\,\mathbf{i} + v(x, y, z)\,\mathbf{j} + w(x, y, z)\,\mathbf{k}$$

(we will usually drop vector symbol above $\vec{F}$, unless needed for clarity).

In this course, we will be concerned with all the smooth[1] vector fields on a three-dimensional subset $\Omega \subset \mathbf{R}^3$. In Nash's problem, think of $\Omega = \mathbf{R}^3 \setminus X$. We call this set of smooth vector fields $VF(\Omega)$. It is a vector space, since you can scale vector fields by multiplying by a constant and you can add vector fields.

Recall the *kernel* of a map $\varphi$ consists of all of the elements it maps to zero, i.e., $\ker \varphi = \{x | \varphi(x) = 0\}$.

**Nash's Problem restated:** Consider $VF(\Omega)$, the space of all smooth vector fields on $\Omega$. Let $V$ be the subspace equal to $\ker \mathrm{curl}$, and let $W$ be the space of all gradients, i.e., $W = \mathrm{Im}\,\mathrm{grad}$. Then what is the dimension of their quotient, i.e.,

$$\dim(V \setminus W) = \dim(\ker \mathrm{curl} \setminus \mathrm{Im}\,\mathrm{grad}) = ?$$

**Exercise 1.2.** The curl of $\nabla g$ is zero, for any gradient. Thus $W \subset V$.

1.1. **Why is this an interesting problem?** This problem relates the local idea of taking a derivative to the global idea of how many 'holes' $\Omega$ has. Recall that a derivative is defined as a limit, so it only depends on the values near a point in question; the operators curl, gradient, and divergence are merely ways of taking derivatives of vector fields and multivariable functions, so they really are local operations. So we ask a question about derivatives, a seemingly local idea, and get an answer that depends on the topology of $\Omega$. If we stretch or deform $\Omega$ without tearing it or making it pass through itself, we obtain the same answer.

This problem relates a local idea (derivatives) to a global idea (the topology of $\Omega$). Problems like that are scintillating! The Gauss-Bonnet Theorem in differential geometry does this for surfaces: it says that if you measure how much a surface is curving at a point (a local idea) and integrate it, the answer only depends on the topology of the surface – how many 'holes' it has (a global idea). The Gauss-Bonnet Theorem forms the crowning achievement of a class on curves & surfaces.

We close with some examples of the answer to Nash's problem.

[1] smooth means all derivatives exist

| $X$ | $\Omega = \mathbf{R}^3 \setminus X$ | $\dim(V \setminus W)$ |
|---|---|---|
| $\emptyset$ | $\mathbf{R}^3$ | 0 |
| point $p$ | $\mathbf{R}^3 \setminus p$ | 0 |
| ball $B$ | $\mathbf{R}^3 \setminus B$ | 0 |
| $z$-axis | $\mathbf{R}^3\setminus$ axis | 1 |
| line $\ell$ | $\mathbf{R}^3 \setminus \ell$ | 1 |
| | solid torus | 1 |
| | solid $n$-holed torus | n |

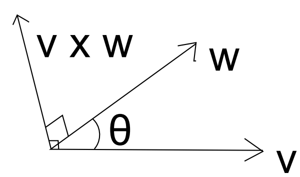
## 2. Vector calculus review I

We review the basics from vector calculus over the next 4 lectures. We will work primarily with vectors $\mathbf{v} \in \mathbf{R}^3$, but occasionally generalize to $\mathbf{R}^n$.

The *inner product*, or *dot product*, of two vectors $\mathbf{v}, \mathbf{w}$ in $\mathbf{R}^n$ is

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + \cdots + v_n w_n = |\mathbf{v}||\mathbf{w}| \cos\theta,$$

where $\theta$ is the angle between them. We note that $\mathbf{v} \cdot \mathbf{w} = 0$ if and only if $\mathbf{v} \perp \mathbf{w}$ – keep in mind that $\vec{0}$ is perpendicular to all vectors. Also, any inner product must be commutative: $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$.




**Theorem 2.1** (Cauchy-Schwarz Inequality)**.** *For* $\mathbf{v}, \mathbf{w}$ *in a vector space endowed with an inner product* $\langle \mathbf{v}, \mathbf{w} \rangle$*,*

$$\langle \mathbf{v}, \mathbf{w} \rangle \leq \|\mathbf{v}\|\|\mathbf{w}\|,$$

*where the norm is the one induced by the inner product,* $\|\mathbf{v}\| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2}$*. Equality holds if and only if* $\mathbf{v}$ *and* $\mathbf{w}$ *are scalar multiples.*

For us, we use the dot product as our inner product; its induced norm is simply the length of $\mathbf{v}$.

The *cross-product* of two vectors in $\mathbf{R}^3$ is defined to be

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \\ &= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{k} \\ &= (v_2 w_3 - v_3 w_2)\, \mathbf{i} - (v_1 w_3 - v_3 w_1)\, \mathbf{j} + (v_1 w_2 - v_3 w_1)\, \mathbf{k} \end{aligned}$$

The cross-product obeys the right-hand rule. The cross product is anti-commutative: $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$. Its length can be expressed by

$$|\mathbf{v} \times \mathbf{w}| = |v||w| \sin\theta.$$

**Definition 2.2.** An *orthogonal matrix* $A$ is one with $A^T = A^{-1}$. The row vectors of any orthogonal matrix forms an orthonormal basis of $\mathbf{R}^n$. An orthonormal basis is a mutually perpendicular linearly independent set of unit vectors spanning $\mathbf{R}^n$.

Any orthogonal matrix respects the dot product $A\mathbf{v} \cdot A\mathbf{w} = \mathbf{v} \cdot \mathbf{w}$; it preserves the cross product, up to a sign change: $A\mathbf{v} \times A\mathbf{w} = \pm\mathbf{v} \times \mathbf{w}$.

**Definition 2.3.** The *triple product* $\mathbf{u} \times \mathbf{v} \cdot \mathbf{w}$ measures the volume of the parallelopiped (3-d slanted box) with sides $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

Permuting $\mathbf{u}, \mathbf{v}, \mathbf{w}$ changes the triple product by $\pm 1$, based on the sign of the permutation:

$$\mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = \mathbf{v} \times \mathbf{w} \cdot \mathbf{u} = \mathbf{w} \times \mathbf{u} \cdot \mathbf{v} = -\mathbf{v} \times \mathbf{u} \cdot \mathbf{w}.$$

**Definition 2.4.** The gradient of a function $f(x_1, \ldots, x_n)$ is the vector field in $\mathbf{R}^n$ given by

$$\nabla f = \langle f_{x_1}, \ldots, f_{x_n} \rangle.$$

**Definition 2.5.** The *level curves* (or *level surfaces*) of a function $f : \mathbf{R}^n \to \mathbf{R}$ consist of all points where $f$ equals the same constant; there's a nonempty level curve for each constant appearing in the range of $f$. The nondegenerate level curves/surfaces of $f$ form $(n-1)$-dimensional subsets of $\mathbf{R}^n$.

For $f(x, y) : \mathbf{R}^2 \to \mathbf{R}$, we can draw its graph $z = f(x, y)$. The level curves represent all the points lying at a constant height. An example of this are contour maps of elevations.

**Fact:** The gradient points orthogonally to all level curves.

**Definition 2.6.** The *directional derivative* of $f$ in direction $\mathbf{u}$, where $\mathbf{u}$ is a unit vector, measures the rate of change of $f$ as we move in the direction of $\mathbf{u}$. It can be written as

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}.$$

**Definition 2.7.** For a surface $S \subset \mathbf{R}^3$, we define its *tangent plane* at point $P$ to be the set of directions that are tangent at $P$ to some curve lying in $S$. A *normal vector* $\mathbf{n}$ to $S$ at $P$ is one orthogonal to its tangent plane.

If $S$ is a level surface of $f(x, y, z)$, then we know $\nabla f$ points orthogonal to it and is thus a normal vector to $S$.

**Example 2.8.** Consider the ellipsoid $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$. Find its tangent plane at $P = (1/3, 4/3, 2)$.

The ellipsoid is a level surface for $f(x, y, z) = x^2 + \frac{y^2}{4} + \frac{z^2}{9}$. We find its gradient to be $\nabla f = \langle 2x, y/2, \frac{2}{9}z \rangle$. Then, $\nabla f(1/3, 4/3, 2) = \langle 2/3, 2/3, 4/9 \rangle$ lies perpendicular to the tangent plane.

A plane in $\mathbf{R}^3$ can be written as $ax + by + cz = d$, where $\langle a, b, c \rangle$ is perpendicular to the plane. In our example, that plane is

$$\tfrac{2}{3}x + \tfrac{2}{3}y + \tfrac{4}{9}z = 2. \qquad \square$$

Finally, we define the divergence and the curl of a vector field.

**Definition 2.9.** The *divergence* of $V = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, where $u, v$, and $w$ are all functions of $x, y, z$ is

$$\operatorname{div} V = \nabla \cdot V = u_x + v_y + w_z.$$

The *curl* of $V = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ is

$$\begin{aligned}\operatorname{curl} V = \nabla \times V = &\det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial x} \\ u & v & w \end{bmatrix} \\ =&(w_y - v_z)\,\mathbf{i} + (u_z - w_x)\,\mathbf{j} + (v_x - u_y)\,\mathbf{k}\end{aligned}$$

n.b., our book (and many European authors) write curl as $\operatorname{rot}$, because $\operatorname{curl} V$ measures the rotation of $V$ around a point, a notion we will make precise in the next two lectures.

## 3. Vector calculus review II: the real meaning of divergence

In this section we try to make the following statement mathematically precise:

> "The divergence of $V$ represents the rate of expansion per unit volume under the flow of the fluid $V$."

**Definition 3.1.** A *homeomorphism* $f : A \to B$ is a one-to-one, onto map between (topological) spaces that is continuous and has continuous inverse.

A *diffeomorphism* is a differentiable homeomorphism with differentiable inverse.

n.b., Many authors require a diffeomorphism to be a smooth homeomorphism; most of our maps will be smooth, while some will only be piecewise smooth, so we can use this latter definition.

A homeomorphism signifies that the two spaces are topologically equivalent.

**Example 3.2.** Provide maps that are continuous but not differentiable, e.g., $f = |x|$, and maps that are differentiable but not smooth: $g = \begin{cases} 0 & x \leq 0 \\ x^2 & x \geq 0 \end{cases}$.

We review the *Jacobian* of a map. For instance if $f = (f_1, f_2, f_3) : \mathbf{R}^3 \to \mathbf{R}^3$, its Jacobian is

$$Jf = \left(\frac{\partial f_i}{\partial x_j}\right) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \dfrac{\partial f_1}{\partial x_3} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \dfrac{\partial f_2}{\partial x_3} \\ \dfrac{\partial f_3}{\partial x_1} & \dfrac{\partial f_3}{\partial x_2} & \dfrac{\partial f_3}{\partial x_3} \end{bmatrix} \tag{3.1}$$

The determinant of the Jacobian measures the stretching effect of the mapping $f$; also $\det Jf > 0$ iff $f$ preserves orientation.

**Example 3.3.** For $f(r,\theta) = (x,y) = (r\cos\theta, r\sin\theta)$ the map from Cartesian coordinates to polar ones, we compute its Jacobian:

$$Jf = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

**Exercise 3.4.** Consider cylindrical coordinates $\{r,\theta,z\}$ and spherical coordinates $\{\rho,\phi,\theta\}$ in $\mathbf{R}^3$. Compute the Jacobians for these coordinate changes from Cartesian coordinates, i.e., find

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} \qquad \text{and} \qquad \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}$$

3.1. **The one-parameter family of diffeomorphisms induced by $V$.** Start with a fluid in the subset $\Omega \subset \mathbf{R}^3$, and let the fluid flow by the smooth vector field $V(x,y,z)$. Let the vector-valued function $\varphi(x,y,z,t) : \Omega \times \mathbf{R} \to \mathbf{R}^3$ represent where the point $(x,y,z)$ has flowed to after $t$ seconds. Since $V$ is smooth, if we let $\Omega$ flow for $t$ seconds, it will go to some subset $\Omega_t$ which is diffeomorphic to $\Omega$.

This means, for any fixed $t$ value, the map $\varphi_t(x,y,z) : \Omega \to \mathbf{R}^3$ is a diffeomorphism of $\Omega$ onto its image. The vector field $V$ can be viewed as a derivative of $\varphi$, and uniquely determines $\varphi$. Precisely,

$$\frac{\partial\varphi}{\partial t} = V\left(\varphi(x,y,z,t)\right) \tag{3.2}$$

Equation (3.2) means, given $V$, we can solve a system of differential equations to find $\varphi$, using the initial condition that $\varphi(x,y,z,0) = (x,y,z)$.

We are now prepared to interpret divergence. The determinant of the Jacobian map of $\varphi_t$ measures to what extent $\Omega$ expands (or contracts) under the map $\varphi_t$. By taking its derivative, we are calculating the rate of expansion. If we do this at $t = 0$, where $\varphi_0$ is the identity map, we are calculating *the instantaneous rate at which $\Omega$ expands by flowing along $V$* – that is precisely what the divergence measures. As a formula, we have

**Theorem 3.5.** *For the construction above and point $p \in \Omega \subset \mathbf{R}^3$,*

$$\nabla \cdot V(p) = \left.\frac{d}{dt}\right|_{t=0} \det J\varphi_t(p) \tag{3.3}$$

We provide a proof of this in the next section.

**Example 3.6.** Consider the vector field $V = x\mathbf{i}$, in which all points move away from the $yz$-plane at a rate equal to their distance from it. We begin by finding $\varphi = \langle\phi_1,\phi_2,\phi_3\rangle$ by solving (3.2):

$$\frac{\partial\varphi}{\partial t} = \left(\frac{\partial\phi_1}{\partial t}, \frac{\partial\phi_2}{\partial t}, \frac{\partial\phi_3}{\partial t}\right) = V(\varphi) = V(\phi_1,\phi_2,\phi_3).$$

This is actually an easy solved system of three ODE's:

$$\left.\begin{aligned} \frac{\partial\phi_1}{\partial t} &= \phi_1 \\ \frac{\partial\phi_2}{\partial t} &= 0 \\ \frac{\partial\phi_3}{\partial t} &= 0 \end{aligned}\right\} \Longrightarrow \begin{aligned} \phi_1 &= c_1e^t \\ \phi_2 &= c_2 \\ \phi_3 &= c_3 \end{aligned} \tag{3.4}$$

Now apply the initial condition that $\varphi(x, y, z, 0) = (x, y, z)$ to find that $(c_1, c_2, c_3) = (x, y, z)$. Thus $\varphi_t = \langle xe^t, y, z\rangle$. This agrees with our expectation that neither the $y$ nor $z$ values can change by flowing only in the **i** direction.

The Jacobian $J\varphi_t$ is

$$\begin{bmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Its determinant is $e^t$ which implies that its derivative at $t = 0$ and hence the divergence of $V$ must both be 1. Of course, it's easier to compute $\nabla \cdot V$ directly.

**Exercise 3.7.** Repeat this example for $V = \mathbf{r}$ and for $V = -y\mathbf{i} + x\mathbf{j} + \mathbf{k}$.

## 4. Vector calculus review III

We continue working with the one-parameter family of diffeomorphisms $\varphi_t$ induced by a vector field $V = \langle u, v, w\rangle$ flowing on a domain. Our goal this time is to determine how the Jacobian $J\varphi_t$ provides a description of both divergence and curl.

Recall that $\{J\varphi_t\}$ is a family (indexed by time $t$) of $3 \times 3$ matrices with $J\varphi_0 = I$. We are going to be concerned with the derivative of $J\varphi_t$; set $B = \left.\frac{dJ\varphi_t}{dt}\right|_{t=0}$. We will call upon 3 useful linear algebra facts. The first one is

**Proposition 4.1.** *Any square matrix $B$ uniquely decomposes into a symmetric part $B_1$ and a skew-symmetric part $B_2$:*

$$\begin{aligned} B &= \quad B_1 \quad + \quad B_2 \\ B &= \frac{B + B^T}{2} + \frac{B - B^T}{2} \end{aligned}.$$

By interchanging time and space derivatives, we obtain

$$\left.\frac{d}{dt}\right|_{t=0} J\varphi_t = J \left.\frac{d\varphi_t}{dt}\right|_{t=0},$$

since the Jacobian takes spatial derivatives. Recall that $d\varphi_t/dt = V(\varphi_t)$, so $d\varphi_t/dt|_{t=0} = V(x, y, z) = V$. Thus,

$$\left.\frac{d}{dt}J\varphi_t\right|_{t=0} = JV = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \tag{4.1}$$

We return now to offer a quick proof of Theorem 3.5. We will need our second linear algebra fact of the day:

The second linear algebra fact is less immediate.

**Proposition 4.2.** *For an $n \times n$ matrix $A(t)$ with $A(0) = I$,*

$$\left.\frac{d}{dt}\right|_{t=0} \det A(t) = \operatorname{tr} \left.\frac{dA}{dt}\right|_{t=0} \tag{4.2}$$

**Exercise 4.3.** Prove Proposition 4.2. This produces a trivial statement when $n = 1$, so start first with an explicit computation for $n = 2$. Then try to use a more sophisticated argument for $n = 3$ and generalize to arbitrary $n$.

*Proof of Theorem 3.5.* Applying (4.2), we see that $\left.\frac{d}{dt}\right|_{t=0} \det J\varphi_t$ equals the trace of $JV$, given in (4.1). But the trace of $JV$ is the divergence $\nabla \cdot V$, which proves our theorem. □

To understand how curl relates to $J\varphi_t$, we turn to our third linear algebra fact:

**Linear Algebra Fact 3:**

- Symmetric matrices are the derivatives of expansion maps along 3 mutually perpendicular axes.
- Skew-symmetric matrices are the derivatives of orthogonal matrices.
- Orthogonal matrices are (products of) rotations (about an axis through the origin) and a reflection.

A rigorous proof of this fact is beyond the scope of our course. An important area in mathematics is the study of *Lie groups* – spaces which have both a group structure and a manifold structure (i.e., locally they look like $\mathbf{R}^n$). Examples of Lie groups are the circle, $\mathbf{R}^n$, and many matrix groups, including $GL(n, \mathbf{R})$ and $O(n)$, the group of orthogonal matrices. The space of tangent vectors at the identity to a Lie group forms its *Lie algebra*. The second bulletpoint above is equivalent to saying the Lie algebra of $O(n)$ consists of the skew-symmetric matrices.

So consider a $3 \times 3$ skew-symmetric matrix $M$:

$$M = \begin{bmatrix} 0 & -c & b \\ c & 0 & a \\ -b & a & 0 \end{bmatrix}$$

It represents the derivative of a rotation in the following way: let $\vec{\omega} = \langle a, b, c \rangle$. Then applying $M$ to a vector $\mathbf{w}$ produces $M\mathbf{w} = \mathbf{u} \times \mathbf{w}$. The direction of $\vec{\omega}$ specifies the axis of rotation, and its magnitude gives the rotational speed. Rotation along $\mathbf{u}$ moves $\mathbf{w}$ in a circle about $\vec{\omega}$; the tangent vector to this rotation is precisely $\mathbf{u} \times \mathbf{w}$. See Figure 4.

So what does this mean to us? The matrix $B_2$, which represents the skew-symmetric part of the derivative of $J\varphi_t$, depicts the derivative of a rotation. Since

$$B_2 = \begin{bmatrix} 0 & u_y - v_x & u_z - w_x \\ v_x - u_y & 0 & v_z - w_y \\ w_x - u_z & w_y - v_z & 0 \end{bmatrix},$$

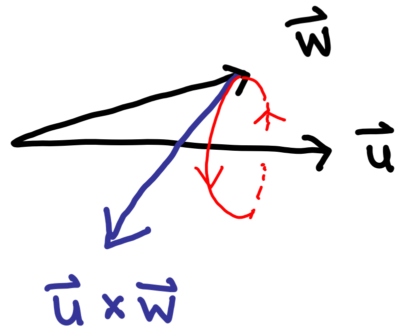



FIGURE 4.1. Rotation along $\mathbf{u}$ moves $\mathbf{w}$ along the red circle about $\mathbf{u}$; the tangent vector to this rotation is precisely $\mathbf{u} \times \mathbf{w}$.

the rotation is about the axis $\omega = \langle w_y - v_z, u_z - w_x, v_x - u_y \rangle$, which is precisely the curl of $V$.

4.1. **Summary: divergence and curl via $J\varphi_t$.** The matrix $\left.\frac{d}{dt}\right|_{t=0} \det J\varphi_t$ decomposes into a symmetric and a skew-symmetric portion. The symmetric part measures the relative rate of expansion via $\varphi_t$; its trace produces $\nabla \cdot V$. The skew-symmetric part depicts the tangent vector to a rotation; the rotation is given by $\nabla \times V$. Hence, both the divergence and curl of a vector field $V$ necessarily appear when considering the Jacobian of its associated family of diffeomorphisms.

4.2. **Leibniz Rules.** Recall the *Leibniz rule*, aka the product rule, from freshman calculus:

$$(fg)' = f'g + fg'.$$

We now consider six different vector versions of the Leibniz Rule. We must consider a new type of operation, $(A \cdot \nabla)B$. For $A = \langle a_1, a_2, a_3 \rangle$, we define $(A \cdot \nabla)B$ to be the vector field

$$(A \cdot \nabla)B = \left( a_1 \frac{\partial}{\partial x} + a_1 \frac{\partial}{\partial x} + a_1 \frac{\partial}{\partial x} \right) B,$$

which represents the vector operation of $A$ on each component function of $B$.[2]

| | | | |
|---|---|---|---|
| (1) | $\nabla(fg)$ | $=$ | $g\nabla(f) + f\nabla(g)$ |
| (2) | $\nabla(A \cdot B)$ | $=$ | $A \times (\nabla \times B) + B \times (\nabla \times A) + (A \cdot \nabla)B + (B \cdot \nabla)A$ |
| (3) | $\nabla \cdot (fA)$ | $=$ | $\nabla f \cdot A + f\nabla \cdot A$ |
| (4) | $\nabla \cdot (A \times B)$ | $=$ | $(\nabla \times A) \cdot B - A \cdot (\nabla \times B)$ |
| (5) | $\nabla \times (fA)$ | $=$ | $\nabla f \times A + f\nabla \times A$ |
| (6) | $\nabla \times (A \times B)$ | $=$ | $-(\nabla \cdot A)B + (\nabla \cdot A)B + (B \cdot \nabla)A - (A \cdot \nabla)B$ |

[2]Indeed, most graduate geometry books write this as the vector field $AB$.

**Remark 4.4.** The last two terms of the last Leibniz Rule, $(B \cdot \nabla)A - (A \cdot \nabla)B$ represent an important concept in higher-level geometry, the *Lie Bracket*. The Lie Bracket is frequently written as

$$[B, A] = BA - AB = (B \cdot \nabla)A - (A \cdot \nabla)B.$$

We mention this only in passing; our use for the Lie bracket is limited to the above Leibniz rule.

4.3. **Second derivatives and the vector Laplacian.** For the three differential operations $\mathrm{div}, \mathrm{grad}, \mathrm{curl}$, there are 9 different combinations which might produce second derivatives. Four of these are nonsensical: $\mathrm{grad} \circ \mathrm{grad}, \mathrm{grad} \circ \mathrm{curl}, \mathrm{div} \circ \mathrm{div}, \mathrm{curl} \circ \mathrm{div}$. The other 5 may be considered as second derivative operators. We have already discussed that $\mathrm{curl} \circ \mathrm{grad} = 0$ and $\mathrm{div} \circ \mathrm{curl} = 0$. We now discuss the other three operations.

**Definition 4.5.** The *Laplacian* of a function $f(x, y, z)$ is

$$\Delta f = f_{xx} + f_{yy} + f_{zz}.$$

The Laplacian is equal to $\mathrm{div} \circ \mathrm{grad}$, i.e., $\Delta f = \nabla \cdot \nabla f$. It is an operator of fundamental importance in many areas of mathematics, including analysis, geometry, differential equations, and applied math.

The last two second derivatives do not have nearly the same recognition; they are $\mathrm{grad} \circ \mathrm{div}$ and $\mathrm{curl} \circ \mathrm{curl}$. However, their difference relates back to the Laplacian.

**Definition 4.6.** For a vector field $V(x, y, z) = \langle u, v, w \rangle$, we define its *vector Laplacian* to be

$$L(V) = \Delta u \mathbf{i} + \Delta v \mathbf{j} + \Delta w \mathbf{k}.$$

**Proposition 4.7.** *The vector Laplacian equals* $\mathrm{grad} \circ \mathrm{div}$ *minus* $\mathrm{curl} \circ \mathrm{curl}$*:*

$$L(V) = \nabla(\nabla \cdot V) - \nabla \times (\nabla \times V)$$

**Exercise 4.8.** Prove the above proposition.

We conclude by summarizing the five well-defined second derivatives:

$$\begin{array}{lrcl}
(1) & \nabla \times \nabla f & = & 0 \\
(2) & \nabla \cdot (\nabla \times V) & = & 0 \\
(3) & \nabla \cdot \nabla f & = & \Delta f \\
(4) & \nabla(\nabla \cdot V) & & \\
(5) & \nabla \times (\nabla \times V) & & \\
 & L(V) & = & \nabla(\nabla \cdot V) - \nabla \times (\nabla \times V)
\end{array}$$

# 5. VECTOR CALCULUS REVIEW IV

5.1. **Line integrals.** We continue our vector calculus review by considering some integrals involving vector fields. The first type are *line integrals*, which measure the flow of a vector field along a curve; the curve is arbitrary and need not be a flowline of the field. For a curve $C$, the line integral of $V$ along $C$ is $\displaystyle\int_C V \cdot d\mathbf{r}$.

To compute a line integral, we parameterize $C$ as $C(t) = (x(t), y(t), z(t))$ where $t \in [a, b]$. We use these to rewrite $V = V(t)$ and then take the dot product with the form $d\mathbf{r} = \langle dx, dy, dz\rangle = \langle x'(t), y'(t), z'(t)\rangle dt$. (The last equality is merely the chain rule in disguise.) The form $d\mathbf{r}$ measures the tangent vector to the curve. So the line integral computes the component of $V$ that is tangent to the curve and integrates it along the length of the curve.

**Definition 5.1.** The line integral of $V$ over a closed curve $C$ is known as its *circulation*. For emphasis, we write the line integral as $\displaystyle\oint V \cdot d\mathbf{r}$ in this case.

**Example 5.2.** Compute the line integral of $V = \langle y, x\rangle$ over the circle $x^2 + y^2 = 4$, traversed counterclockwise.

We begin by parametrizing the circle as $x(t) = 2\cos t,\ y(t) = 2\sin t$. Then, the form $d\mathbf{r} = 2\langle -\sin t, \cos t\rangle dt$, and $V = 2\langle \sin t, \cos t\rangle$. Thus,

$$\begin{aligned}\oint V \cdot d\mathbf{r} &= \int_0^{2\pi} 4(\cos^2 t - \sin^2 t)\ dt \\ &= \int_0^{2\pi} 4\cos(2t)\ dt \qquad \text{(using a trig identity)} \\ &= 2\ \sin(2t)\big|_0^{2\pi} \\ &= 0 \qquad \square\end{aligned}$$

**Proposition 5.3.** *1. The circulation of a gradient over a closed curve must be zero, i.e.,* $\displaystyle\oint_C \nabla f \cdot d\mathbf{r} = 0$.
*2. For an arbitrary curve $C_1$ from $p$ to $q$, the line integral of a gradient is the difference of the values at the endpoints:* $\displaystyle\int_{C_1} \nabla f \cdot d\mathbf{r} = f(q) - f(p)$.
*3. Line integrals are independent of our choice of parameterizing $C$.*

*Proof.* To understand why, recall that $\nabla f \cdot \mathbf{u}$ was the directional derivative of $f$ in the direction of $\mathbf{u}$. So $\nabla f \cdot d\mathbf{r}$ measures the rate of change of $f$ as we move tangent to the curve; by integrating, we obtain the difference in $f$'s values from $p$ to $q$.

The first statement is a consequence of the second. The third is merely a change of variables argument. $\square$

A useful, visual example is where $f(x, y)$ represents the height on a mountain and maybe $C$ depicts a hiking trail. The line integral along $C$ measures your net change in elevation as you traverse the trail. If the trail ends where it starts, i.e., $C$ is closed, then your net elevation change is zero.

The vector field in Example 5.2 is the gradient of $f = xy$, so its circulation must be zero.

5.2. **Surface integrals.** Now we turn to surface integrals. Given a surface $S$, we will want (rarely) to integrate functions $g$ over $S$ and (often) to measure the flux of a vector field over $S$. We write these integrals as

$$\int_S g\, dA \qquad \text{and} \int_S (V \cdot \hat{\mathbf{n}})\, dA,$$

where $dA$ represents the area form on $S$. n.b., we will not write multiple integral signs even though the integral over $S$ is a double integral.

**Definition 5.4.** The *flux* $\int_S (V \cdot \hat{\mathbf{n}})\, dA$ of a vector field $V$ over an oriented surface $S$ measures the component of $V$ that is flowing across $S$ (as opposed to flowing tangent to $S$). Here $\mathbf{n}$ is the unique unit normal vector to $S$ which agrees with its orientation.

To compute a surface integral, begin by parameterizing the surface in terms of variables $u, v$; that is find a map $f$ from a subset of the $uv$-plane into $\mathbf{R}^3$ whose image is $S$. (We can break $S$ into pieces if necessary, e.g., if it is the outside of a cube.) Then a normal vector is given by the cross product $N = f_u \times f_v$. To find $\hat{\mathbf{n}}$, normalize $N$ and pick the appropriate orientation, i.e.,

$$\hat{\mathbf{n}} = \pm \frac{f_u \times f_v}{|f_u \times f_v|}.$$

Now to compute the area form $dA$. If we were integrating a region in the $xy$-plane, it would just be $dxdy$. For other parametrizations, we must measure the amount that they stretch area; this stretch is precisely given by $|f_u \times f_v|$, so $dA = |f_u \times f_v|\, dudv$. Thus, we conclude that the flux integrand is

$$V \cdot \hat{\mathbf{n}})\, dA = V \cdot \pm \frac{f_u \times f_v}{|f_u \times f_v|} |f_u \times f_v|\, dudv = V \cdot \pm (f_u \times f_v)\, dudv$$

**Remark 5.5.** 1. Surfaces with boundary acquire their orientation via the right-hand rule from their boundary curve. For example, the unit disk in the $xy$-plane, when bounded by the circle oriented counterclockwise, acquires an orientation so that $\hat{\mathbf{n}}$ points up. (Curl your right hand along the circle counterclockwise; your thumb will point up.) If the circle is oriented clockwise, then $\hat{\mathbf{n}}$ will point down.

2. For surfaces without boundary, such as the sphere or torus, we may assume that they are oriented so that $\hat{\mathbf{n}}$ points out. We orient the plane so $\hat{\mathbf{n}}$ points up.

**Example 5.6.** Parameterize the unit sphere by spherical coordinates, and then calculate the flux of $V = xy\mathbf{i} + xz\mathbf{k}$.

Spherical coordinates state that $x = \rho \cos\theta \sin\phi$, $y = \rho \sin\theta \sin\phi$, and $z = \rho\cos\phi$, where $\rho \in [0, \infty), \phi \in [0, \pi], \theta \in [0, 2\pi]$. For the unit sphere, $\rho = 1$. Thus our surface parameterization is

$$f(\phi, \theta) = \langle \cos\theta \sin\phi,\ \sin\theta\sin\phi,\ \cos\phi \rangle.$$

We calculate that $f_\phi \times f_\theta = \sin\phi\, \langle x, y, z\rangle$. Thus $\hat{\mathbf{n}} = \pm \dfrac{\sin\phi}{|\sin\phi|}\ \langle x, y, z\rangle = \pm\langle x, y, z\rangle$. Note that $\sin\phi$ is always nonnegative on its domain. We want $\hat{\mathbf{n}}$ to point outward, so we pick $+\langle x, y, z\rangle$.

To calculate the flux, we perform the double integral

$$\int_S V \cdot \hat{\mathbf{n}} = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} V \cdot |f_\phi \times f_\theta| \, d\phi d\theta.$$

This integral seems much worse than it is; symmetry allows a lot of things to cancel or integrate to zero. In fact, the whole integral equals 0.

Geometrically, we can see this by looking at the flux of the term $xy\mathbf{i}$. On the front of the sphere, this will be flowing out when $x, y$ have the same sign, and flowing in when they don't. These contributions precisely cancel each other. On the back of the sphere, this flows in when $x, y$ have the same sign, and flows out when they don't. Again the contributions cancel. The same behavior is similarly true for $xz\mathbf{k}$. This should fully convince you that the flux must be zero. $\square$

5.3. **Integral Theorems.** We close with the two crowning integral theorems of vector calculus, which you will explore in homework.

**Theorem 5.7** (Divergence Theorem)**.** *Let $\Omega$ be a compact (i.e., a closed and bounded) three-dimensional subset of $\mathbf{R}^3$ with piecewise smooth boundary $\partial\Omega$. Let $\hat{\mathbf{n}}$ be the unit outward normal vector to the boundary , and let $V$ be a smooth vector field on $\Omega$. Then,*

$$\int_\Omega \nabla \cdot V \; dvol = \int_{\partial\Omega} V \cdot \hat{\mathbf{n}} \; dA \tag{5.1}$$

**Theorem 5.8** (Stokes' Theorem)**.** *Let $S$ be a compact (i.e., a closed and bounded) orientable surface with piecewise smooth boundary $\partial S$. Let $\hat{\mathbf{n}}$ be the unit outward normal vector to its boundary, and let $V$ be a smooth vector field on $S$. Then,*

$$\int_S \nabla \times V \cdot \hat{\mathbf{n}} \; dA = \int_{\partial S} V \cdot d\mathbf{r} \tag{5.2}$$

We note that $\partial\Omega$ might consist of multiple surfaces; in this case we break up the surface integral into one per boundary surface and sum the results. Similarly, $\partial S$ might consist of multiple curves; in this case we break up the line integral into one per boundary curve and sum the results.